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1. Buffon's Problems

Buffon's experiments are very old and famous random experiments, named after Compte De Buffon. These experiments are considered to be among the first problems in geometric probability.

Buffon's Coin Experiment

Buffon's coin experiment consists of dropping a coin randomly on a floor covered with identically shaped tiles. The event of interest is that the coin crosses a crack between tiles. We will model Buffon's coin problem with square tiles of side length 1 (assuming the side length is 1 is equivalent to taking the side length as the unit of measurement).

Assumptions

First, let us define the experiment mathematically. As usual, we will idealize the physical objects by assuming that the coin is a perfect circle with radius r and that the cracks between tiles are line segments. A natural way to describe the outcome of the experiment is to record the center of the coin relative to the center of the tile where the coin happens to fall. More precisely, we will construct coordinate axes so that the tile where the coin falls occupies the square $S = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$

Now when the coin is tossed, we will denote the center of the coin by $(X, Y) \in S$ so that S is our sample space and X and Y are our basic random variables. Finally, we will assume that $r < \frac{1}{2}$ so that it is at least possible for the coin to fall inside the square without touching a crack.
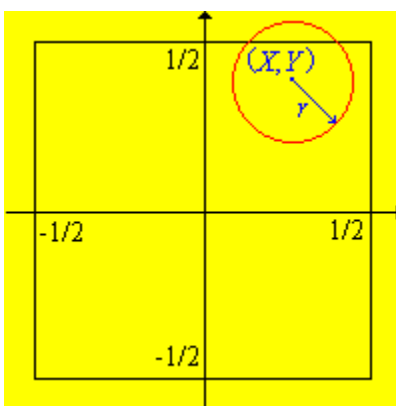
Next, we need to define an appropriate probability measure that describes our basic random vector (X, Y) If the coin falls "randomly" on the floor, then it is natural to assume that (X, Y) is uniformly distributed on S. By definition, this means that

$$\mathbb{P}((X, Y) \in A) = \frac{\text{area}(A)}{\text{area}(S)}, \quad A \subseteq S$$

1. Run Buffon's coin experiment with the default settings. Watch how the points seem to fill the sample space S in a uniform manner.
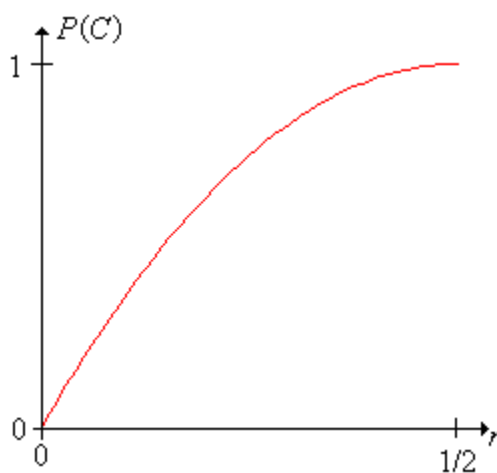
The Probability of a Crack Crossing

Our main interest is in the event C that the coin crosses a crack. However, it turns out to be easier to describe the complementary event that the coin does not cross a crack.

2. Show that $C^c = \left\{ r - \frac{1}{2} < X < \frac{1}{2} - r,\ r - \frac{1}{2} < Y < \frac{1}{2} - r \right\}$

3. Use the result of Exercise 2 to show that $\mathbb{P}(C) = 1 - (1 - 2r)^2$

4. Show that $\mathbb{P}(C)$ as a function of r, has the graph given below:

5. In Buffon's coin experiment, change the radius with the scroll bar and watch how the events C and C^c change. Run the experiment with various values of r and compare the physical experiment with the points in the scatterplot. Note the apparent convergence of relative frequency of C to the probability of C.

The convergence of the relative frequency of an event (as the experiment is repeated) to the probability of the event is a special case of the law of large numbers.

6. Solve Buffon's coin problem with rectangular tiles that have height h and width w.

7. Solve Buffon's coin problem with equilateral triangular tiles that have side length 1.

8. Show how to simulate the center of the coin (X, Y) in Buffon's coin experiment using random numbers.

Buffon's Needle Problem

Buffon's needle experiment consists of dropping a needle on a hardwood floor. The main event of interest is that the needle crosses a crack between floorboards. Strangely enough, the probability of this event leads to

a statistical estimate of the number π!

Assumptions

Our first step is to define the experiment mathematically. Again we idealize the physical objects by assuming that the floorboards are uniform and that each has width 1. We will also assume that the needle has length $L < 1$ so that the needle cannot cross more than one crack. Finally, we assume that the cracks between the floorboards and the needle are line segments.

When the needle is dropped, we want to record its orientation relative to the floorboard cracks. One way to do this is to record the angle X that the top half of the needle makes with the line through the center of the needle, parallel to the floorboards, and the distance Y from the center of the needle to the bottom crack. These will be the basic random variables of our experiment, and thus the sample space of the experiment is

$$S = [0, \pi) \times [0, 1) = \{(x, y) : 0 \leq x < \pi \text{ and } 0 \leq y < 1\}$$
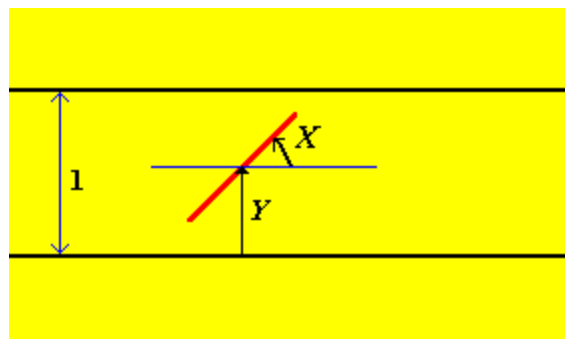
Again, our main modeling assumption is that the needle is tossed "randomly" on the floor. Thus, a reasonable mathematical assumption might be that the basic random vector (X, Y) is uniformly distributed over the sample space. By definition, this means that

$$\mathbb{P}((X, Y) \in A) = \frac{\text{area}(A)}{\text{area}(S)}, \quad A \subseteq S$$

9. Run Buffon's needle experiment with the default settings and watch the outcomes being plotted in the sample space. Note how the points in the scatterplot seem to fill the sample space S in a uniform way.
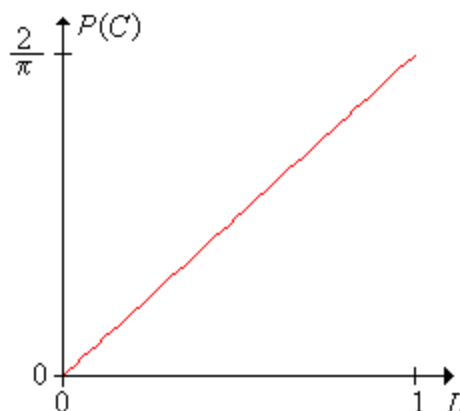
The Probability of a Crack Crossing

Our main interest is in the event C that the needle crosses a crack between the floorboards.

10. Use trigonometry to show that C can be written in terms of the basic angle and distance variables as follows:

$$C = \left\{ Y < \frac{L}{2} \sin(X) \right\} \cup \left\{ Y > 1 - \frac{L}{2} \sin(X) \right\}$$

11. Show that area$(C) = 2L$ and therefore $\mathbb{P}(C) = \frac{2L}{\pi}$

⊞ 12. Show that $\mathbb{P}(C)$, as a function of L, has the following graph:

The curves $y = \frac{L}{2}\sin(x)$ and $y = 1 - \frac{L}{2}\sin(x)$ are shown in blue in the scatterplot, and hence event C is the union of the regions below the lower curve and above the upper curve. Thus, the needle crosses a crack precisely when a point falls in this region.

⊞ 13. In the Buffon's needle experiment, vary the needle length L with the scroll bar and watch how the events C and C^c change. Run the experiment with various values of L and compare the physical experiment with the points in the scatterplot. Note the apparent convergence of relative frequency of C to the probability of C.

The convergence of the relative frequency of an event (as the experiment is repeated) to the probability of the event is a special case of the law of large numbers.

⊞ 14. Find the probabilities of the following events in Buffon's needle experiment. In each case, sketch the event as a subset of the sample space.

  a. $\left\{ 0 < X < \frac{\pi}{2}, \, 0 < Y < \frac{1}{3} \right\}$
  b. $\left\{ \frac{1}{4} < Y < \frac{2}{3} \right\}$
  c. $\{ X < Y \}$
  d. $\{ X + Y < 2 \}$

The Estimate of π

Suppose that we run Buffon's needle experiment a large number of times. By the law of large numbers, the *proportion* of crack crossings should be about the same as the *probability* of a crack crossing. More precisely, we will denote the number of crack crossings in the first n runs by N_n. Note that N_n is a random variable for the compound experiment that consists of n replications of the basic needle experiment. Thus, if n is large, we should have

$$\frac{N_n}{n} \approx \frac{2L}{\pi} \quad \text{and hence} \quad \pi \approx \frac{2Ln}{N_n}$$

This is Buffon's famous estimate of π. In the simulation, this estimate is computed on each run and shown numerically in the second table and visually in the bar graph.

15. Run the Buffon's needle experiment with needle lengths $L \in \{0.3, 0.5, 0.7, 1\}$. In each case, watch the estimate of π as the simulation runs.

Let us analyze the estimation problem more carefully. On each run j we have the indicator variable

$$I_j = \begin{cases} 1, & \text{the needle crosses a crack on run } j \\ 0, & \text{the needle does not cross a crack on run } j \end{cases}$$

These indicator variables are independent, and identically distributed, since we are assuming independent replications of the experiment. Thus, the sequence forms a Bernoulli trials process.

16. Show that the number of crack crossings in the first n runs of the experiment is
$$N_n = \sum_{j=1}^{n} I_j$$

17. Use the result of Exercise 15 to show that the number of crack crossings in the first n runs has the binomial distribution with parameters n and $p = \frac{2L}{\pi}$

18. Use the result of Exercise 16 to show that the mean and variance of the number of crack crossings are

   a. $\mathbb{E}(N_n) = \frac{2Ln}{\pi}$
   b. $\text{var}(N_n) = \frac{2Ln}{\pi}\left(1 - \frac{2L}{\pi}\right)$

19. Use the strong law of large numbers to show that with probability 1,

   a. $\frac{N_n}{2Ln} \to \frac{1}{\pi}$ as $n \to \infty$
   b. $\frac{2Ln}{N_n} \to \pi$ as $n \to \infty$

Thus, we have two basic estimators:

- $\frac{N_n}{2Ln}$ as an estimator of $\frac{1}{\pi}$
- $\frac{2Ln}{N_n}$ as an estimator of π.

Properties

The estimator of $\frac{1}{\pi}$ has several important statistical properties. First, it is *unbiased* since the expected value of the estimator is the parameter being estimated:

⊞ 20. Use Exercise 17 and properties of expected value to show that
$$\mathbb{E}\left(\frac{N_n}{2Ln}\right) = \frac{1}{\pi}$$

Since this estimator is unbiased, the variance gives the mean square error:

$$\text{var}\left(\frac{N_n}{2Ln}\right) = \mathbb{E}\left(\left(\frac{N_n}{2Ln} - \frac{1}{\pi}\right)^2\right)$$

⊞ 21. Use Exercise 17 and properties of variance to show that
$$\text{var}\left(\frac{N_n}{2Ln}\right) = \frac{\pi - 2L}{2Ln\pi^2}$$

⊞ 22. Show that the variance in Exercise 20 is a decreasing function of the needle length L.

Thus, the estimator of $\frac{1}{\pi}$ improves as the needle length increases. On the other hand, the estimator of π is biased; it tends to overestimate π:

⊞ 23. Use Jensen's inequality to show that
$$\mathbb{E}\left(\frac{2Ln}{N_n}\right) \geq \pi$$

The estimator of π also tends to improve as the needle length increases. This is not easy to see mathematically. However, you can see it empirically.

⊞ 24. In the Buffon's needle experiment, set the update frequency to 100. Run the simulation 5000 times each with $L = 0.3$, $L = 0.5$, $L = 0.7$, and $L = 1$, Note how well the estimator seems to work in each case.

Finally, we should note that as a practical matter, Buffon's needle experiment is not a very efficient method of approximating π. According to Richard Durrett, to estimate π to four decimal places with $L = \frac{1}{2}$ would require about 100 million tosses!

⊞ 25. Run the Buffon's needle experiment with an update frequency of 100 until the estimates of π seem to be consistently correct to two decimal places. Note the number of runs required. Try this for needle lengths $L = 0.3$, $L = 0.5$, $L = 0.7$, and $L = 1$ and compare the results.

⊞ 26. Show how to simulate the angle X and distance Y in Buffon's needle experiment using random numbers.

Notes

Buffon's needle problem is essentially solved by **Monte-Carlo integration**. In general, **Monte-Carlo methods** use statistical sampling to approximate the solutions of problems that are difficult to solve analytically. The modern theory of Monte-Carlo methods began with Stanislaw Ulam, who used the methods on problems associated with the development of the hydrogen bomb.

The original needle problem has been extended in many ways, starting with Simon Laplace who considered a floor with rectangular tiles. Indeed, variations on the problem are active research problems even today.

Neil Weiss has pointed out that our computer simulation of Buffon's needle experiment is circular, in the sense the program assumes knowledge of π (you can see this from the result of Exercise 26).

⊞ 27. Try to write a computer algorithm for Buffon's needle problem, without assuming the value of π or any other transcendental numbers.